Performance Modeling for All-Optical Photonic Switches Based on the Vertical Stacking of Banyan Network Structures

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***Abstract*—The scheme of vertical stacking has caught much interest in the design of ultra high-speed communication switches for the past a few years. In particular, the scheme can greatly facilitate the effort of constructing all-optical photonic switches in the event that a Banyan network structure is adopted. A vertically stacked photonic Banyan (VSPB) network can preserve the good properties of the Banyan network structures, such as the small depth and absolute loss uniformity; on the other hand, it introduces a significant increase in the hardware cost. Extensive research efforts have been addressed in determining the minimum number of stacked copies (planes) required for a nonblocking VSPB network. However, very few of them focused on the performance of the VSPB networks in terms of blocking probability. Therefore, in this paper, we study the blocking behavior of the VSPB networks and propose a corresponding analytical model under the random routing strategy. The proposed analytical model is designed to fully explore the property of symmetry in Banyan network structures, and can calculate the blocking probability of a VSPB network stage by stage in a recursive manner such that the combinatorial explosion problem is avoided. To verify the proposed model, we conduct extensive simulations, in which the results indicate that our model can accurately describe the blocking behavior of VSPB networks under the random routing strategy and it agrees with the conditions of strictly nonblocking VSPB networks. We find that the proposed analytical model can deeply investigate into the inherent relationship between blocking probability and network hardware cost in terms of the number of planes; as a result, a quantitative guidance for initiating a graceful compromise between blocking probability and hardware cost can be developed based on the analytical model. Our analysis results also show that the hardware cost of a VSPB network can be dramatically reduced by simply allowing a negligible nonzero blocking probability in most of the practical cases. This fact will solidly contribute to the network switch architecture design and enable more practical applications of VSPB networks.**

***Index Terms*—Banyan networks, blocking probability, optical switches, random routing, vertical stacking.**

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I. Introduction

WITH THE RAPID growth of the Internet, the bandwidth demand for multimedia applications will exponentially increase. The use of all-optical photonic networks based on the wavelength-division-multiplexing (WDM) technology is considered as a promising candidate to meet this demand. Instead of adopting ring or star architectures, WDM networks with mesh topology have recently caught much more interest than ever due to the mesh-in-nature Internet backbones that are considered more capacity-efficient and survivable. In WDM mesh networks, all-optical photonic switches play a key role in the transportation plane, which are embedded with the intelligence of routing and signaling along with the enabling technology in handling complex mesh topologies and large numbers of inputs with different wavelengths, particularly at switching hubs that deal with a large amount of optical flows. A large-scale photonic switch is usually composed of numerous basic switching elements (SEs) grouped in multiple stages along with the optical links arranged in a specified interconnection pattern between adjacent stages, which is expected to have the capability of switching huge optical data with an ultrahigh speed. The basic SEs and the interconnecting optical links in a photonic switching device will form a network such that the optical flows at inputs can be transported to outputs of the switch. We refer to the interconnection pattern, the basic SEs, and the input/output ports as the *network* of the all-optical photonic switch.

The most mature technology for implementing the basic 2×2 SEs in networks of photonic switches is directional-couplers (DCs) [1], [2]. DC is an electro-optical device implemented by manufacturing two waveguides close to each other. The cross (bar) state of a DC is created by applying a suitable voltage (no voltage) to it. DC can pass multiple-wavelength optical signals and this makes it ideal for optical cross-connects (OXCs). However, DC suffers from an intrinsic crosstalk problem [1], [3], in which a portion of optical power in one waveguide of a DC will be coupled into the other unintended waveguide when two optical signals pass through the DC at the same time. This undesirable coupling effect is called the first-order crosstalk. This first-order crosstalk will propagate downstream stage by stage, leading to a higher order crosstalk in each downstream stage with a decreasing magnitude. Due to the stringent bit-error rate requirement of fiber optics, crosstalk elimination has become an important issue for improving the signal-to-noise ratio of the optical flow transmission. A cost-effective solution to the crosstalk problem is to guarantee that only one signal passes through a DC at a time so that the first-order crosstalk can be eliminated.

The topology of Banyan type networks [4]–[7] is a very popular structure for building communication switches. This class of networks is characterized by having a simple switch setting ability (self-routing), a small depth, and an identical number of SEs along any path between an input-output pair. These characteristics have made Banyan topologies promising for serving as the networks of DC-based photonic switches because the amount of loss and attenuation experienced by an input optical signal is proportional to the number of DCs traversed by the optical signal. However, with the Banyan topology only a unique path can be found from each network input to each network output, in which the network is degraded as a blocking one. To deal with this situation, vertical stacking multiple copies of photonic Banyan network structures is an effective approach for keeping the whole network nonblocking [8]. This type of networks is called vertically stacked photonic Banyan (VSPB) networks, which preserves the nonblocking characteristic, while neither increasing the number of stages nor sacrificing the loss uniformity property originally possessed by the Banyan network structures.

In this paper, we will focus on the performance modeling for VSPB networks that are free of first-order crosstalk in all SEs in the networks. (This quality is referred to as *crosstalk-free* in the following context.) The consideration of the crosstalk-free constraint distinguishes the analysis on photonic switching networks from that for electronic ones. Conventionally, blocking happens when two connections intend to use the same link, which is referred to as *link-blocking*. Obviously, all signals passing through a network should follow link-disjoint paths in transmission to avoid link-blocking. In VSPB networks, however, there is another type of blocking. If adding the connection causes some paths including the new one to violate the crosstalk-free constraint, the connection cannot be added even if the path is available. We refer to this second type of blocking as *crosstalk-blocking*. Since the crosstalk-free constraint requires any two optical signals never sharing an SE in transmission (i.e., they should be node-disjoint in transmission), the consideration of the crosstalk-blocking will further increase the overall blocking probability rather than only considering the link-blocking case.

The VSPB networks have been extensively studied in the literature [9]–[12]. Previous work has mainly focused on determining the minimum number of planes required to achieve a nonblocking (crosstalk-free) VSPB network. These studies showed that the adoption of the vertical stacking scheme, although is attractive, will impose a prohibitively high hardware cost.

A probabilistic analysis on the blocking behavior for a network that does not rigidly meet the nonblocking requirement is an effective approach to initiating a compromise between the hardware cost and the blocking performance. Lee [13] and Jacobaeus [14] have developed two well-known probabilistic models, respectively, for analyzing the blocking behavior of Clos networks [15]. With similar approaches to those proposed by Lee and Jacobaeus, a number of studies have been conducted on the performance analysis for Banyan networks [5], [7], [16], [17], which, however, are the probabilistic results only for electronic networks. In other words, only link-blocking, but crosstalk-blocking, was addressed in their studies. To our best knowledge, there has been little known on the performance analysis for VSPB networks in terms of blocking probability. Although a framework was developed in [18] to estimate both the upper and lower bounds on blocking probability of a VSPB network, these bounds may not be used to accurately model the real blocking probability of a VSPB network. It is clear that no analytical result has been reported to model the real blocking probability for the all-optical photonic switches based on vertical stacking of Banyan network structures.

In this paper, we conduct a comprehensive study of actual blocking probability in a VSPB network whose hardware cost (number of stacked planes) may not rigidly follow the nonblocking requirement. The main contributions of our work are the following.

- We extend the probabilistic methods used for the analysis of electronic networks where only link-blocking is concerned such that the performance in terms of blocking probability in a VSPB network can be analyzed, where the crosstalk-blocking and link-blocking are jointly considered under random routing.
- The analytical model calculates the blocking probability of a VSPB network efficiently by fully exploring the property of symmetry in Banyan networks as well as the stage-by-stage recursive characteristic in calculation to avoid the combinatorial explosion problem. The model can demonstrate inherent relationship between the blocking probability and network hardware cost in terms of the number of vertically stacked planes. It is consistent with the condition of strictly nonblocking, and it can accurately describe the blocking behavior of VSPB networks under random routing.
- A network simulator is developed to verify our analytical model with extensive simulation. It is observed that the simulation results well match the estimation given by the proposed model.

Our model can guide network designers to determine the effects of reduction in the number of planes on the overall blocking behavior in VSPB networks under random routing as well as to initiate a compromise between the hardware cost and the blocking probability by manipulating the number of planes. An important conclusion drawn from our work that is significant to practical designs and applications of VSPB networks is that we can dramatically reduce the hardware cost of a VSPB network by allowing a predictable and negligibly small blocking probability.

The rest of the paper is organized as follows. Section II provides definitions that will facilitate the further discussions. Section III introduces the proposed analytical model for VSPB networks. Section IV presents the simulation results, which is compared with the theoretical ones estimated by our analytical model. Section V concludes the paper.

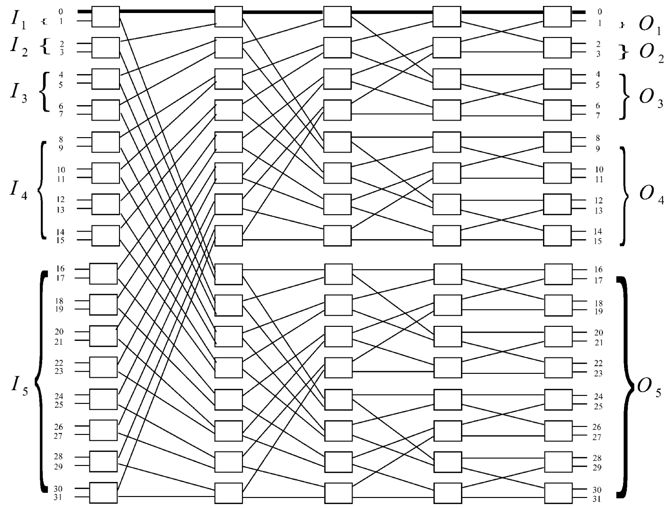

Fig. 1. A 32 × 32 Banyan network (odd number of stages).

II. Preliminaries

A typical $N \times N$ Banyan network consists of $\log N$[1] stages, each of which contains $N/2$ 2×2 basic SEs. The link connection between each pair of adjacent stages is arranged in such a way that a butterfly interconnection pattern is recursively applied as shown in Fig. 1. In this paper, we consider the Banyan networks that support *one-to-one* communication. The blocking probability of a VSPB network is defined as the probability that a *feasible connection request* is blocked, where a feasible connection request is defined as a connection request between an idle input port and an idle output port of the network. Due to the symmetry of a Banyan network, all paths have the same property in terms of blocking. Without loss of generality, we will focus on the path between the first input and the first output (which is termed as the *tagged path* in the following context). All the SEs on the tagged path are called *tagged SEs*. The stages of SE are numbered from left (stage 1) to right (stage $\log N$). For the tagged path, the input intersecting set $I_i = \{2^{i-1}, 2^{i-1}+1, \ldots, 2^i - 1\}$ at stage i is defined as the set of all inputs that intersect a tagged SE at stage i, and the output intersecting set $O_i = \{2^{i-1}, 2^{i-1}+1, \ldots, 2^i - 1\}$ associated with stage i is defined as the set of all outputs that intersect a tagged SE at stage $\log N - i + 1$.

To route a feasible connection through a VSPB network, a routing algorithm must be adopted to find a path through the network. In this paper, we consider a random-routing algorithm. Based on the random-routing algorithm, request router randomly chooses one of the planes that can be used by a feasible request to establish the path for the request. If no plane can satisfy the feasible connection request, the connection request is blocked.

To facilitate our analysis, we will follow the same assumption as in [13] and [14] on probabilistic analysis of multistage interconnection networks (MINs): the correlation among signals arriving at input (output) ports will be neglected, which results in a fact that the status (busy or idle) of each individual input (output) port in the network is independent. This assumption matches the practical situation in that the networks are becoming larger in size so that the increasingly more complex interconnections and huge amount of data can be handled at once. In such a circumstance, instead of being fixed with some extent of mutual correlation, the communication patterns of the inputs (outputs) signals to a photonic switch is getting statistically random, and the correlation among signals at inputs (outputs) becomes approximately negligible.

[1]In this paper, log means logarithm to the base 2.

III. Analytical Model for Random Routing

In this section, the analytical models on blocking probability of $\mathrm{VSPB}(N,m)$ will be introduced for the cases of even and odd numbers of stages, where $\mathrm{VSPB}(N,m)$ denotes an $N \times N$ VSPB network with m stacked copies (planes) of an $N \times N$ Banyan network. When $\log N$ is even, the tagged path is composed of two symmetric halves, each of which has $(1/2)\log N$ stages. The connections destined for output intersecting sets in the second half have the same properties as those originated from the input intersecting sets in the first half. When $\log N$ is odd, the tagged path is also composed of two symmetric halves, each of which has $(\log N + 1)/2$ stages with the central stage of SE being overlapped.

In general, the analysis of blocking probability of $\mathrm{VSPB}(N,m)$ is complex and difficult because some connections conflicting with the tagged path may share planes under the crosstalk-free constraint and others may not. The dependency and restriction among these connections lead to the combinatorial explosion problem. In the rest of this section, we will develop a novel approach to recursively computing the blocking probability of VSPB.

A. Blocking Probability When $\log N$ is Odd

In the event that $\log N$ is odd (please refer to Fig. 1), we define input set I_{odd} and output set O_{odd} as

$$I_{\text{odd}} = \bigcup_{i=1}^{(\log N+1)/2} I_i, \quad O_{\text{odd}} = \bigcup_{i=1}^{(\log N+1)/2} O_i.$$

Note that under the crosstalk-free constraint, a plane of a $\mathrm{VSPB}(N,m)$ network can be blocked in the first half of the tagged SEs (from stage 1 to stage $(1/2)(\log N + 1)$) or/and in the second half of the tagged SEs (from stage $(1/2)(\log N + 1)$ to stage $\log N$). Then, the maximum number of conflicts with the first half of the tagged SEs and second half of the tagged SEs are determined by the connections from set I_{odd} and the connections destined for set O_{odd}, respectively. If a plane of a $\mathrm{VSPB}(N,m)$ network is blocked in both the first and second halves of tagged SEs (i.e., a plane blocked in the first half of tagged SEs shares the same plane blocked in the second half of tagged SEs), this blocked plane is said to be overlapped. Let n_i denotes the number of connections passing through the ith tagged SE, and $b^{i,i+1,\ldots,i+k}$ denotes the number of planes blocked by the connections passing through the ith tagged SE, the connections passing through the $(i+1)$th tagged SE,…, and the connections passing through the $(i+k)$th tagged SE. If we use $X = a$ to denote the event that a random variable X takes value a and use $\Pr(X = a)$ to denote the probability that event

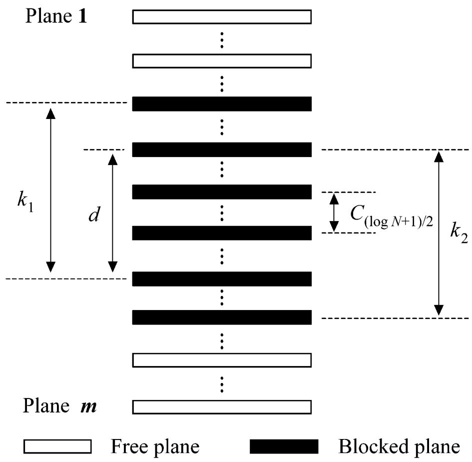

Fig. 2. $\mathrm{VSPB}(N,m)$ with k_1 planes blocked in the first half of the tagged SEs, k_2 planes blocked in the second half of tagged SEs, among them a total of d blocked planes are overlapped.

$X = a$ happens, then we can establish the following lemma concerning the overlapped planes.

Lemma 1: Given events $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$ and $b^{(1/2)(\log N+1),\ldots,\log N} = k_2$ in a $\mathrm{VSPB}(N,m)$ network when $\log N$ is odd ($c_{(1/2)(\log N+1)} \le k_1, k_2 \le m-1$), the probability that there are d blocked planes overlapped is given by

$$\begin{aligned}\Pr\Big(&d \text{ blocked planes overlapped} \,\Big|\, n_{(1/2)(\log N+1)} \\ &= c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\ &= k_1, b^{(1/2)(\log N+1),\ldots,\log N = k_2}\Big) \\ &= \frac{\binom{k_1-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_1}{k_2-d}}{\binom{m-c_{(1/2)(\log N+1)}}{k_2-c_{(1/2)(\log N+1)}}} \\ &= \frac{\binom{k_2-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_2}{k_1-d}}{\binom{m-c_{(1/2)(\log N+1)}}{k_1-c_{(1/2)(\log N+1)}}}. \end{aligned} \tag{1}$$

Proof: Given events $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$ and $b^{(1/2)(\log N+1),\ldots,\log N} = k_2$ in a $\mathrm{VSPB}(N,m)$ under random routing, we have a total of $\binom{m}{k_1}\binom{m-c_{(1/2)(\log N+1)}}{k_2-c_{(1/2)(\log N+1)}}$ ways to select k_1 blocked planes corresponding to the first half of the tagged SEs and k_2 blocked planes corresponding to the second half of the tagged SEs, among which a total of d overlapped planes can be constructed as follows (please refer to Fig. 2). First, we have $\binom{m}{k_1}$ ways to select k_1 blocked planes out of m stacked copies corresponding to the first half of the tagged SEs, along with $\binom{k_1-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}$ ways to select a total of d overlapped planes that are blocked in both the first and second halves of tagged SEs. Then, we have $\binom{m-k_1}{k_2-d}$ choices to select k_2-d planes blocked in the second half of the tagged SEs out of the total $m-k_1$ planes. Therefore, given events $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$ and $b^{(1/2)(\log N+1),\ldots,\log N} = k_2$ in a $\mathrm{VSPB}(N,m)$ network ($c_{(1/2)(\log N+1)} \le k_1, k_2 \le m-1$),the probability that there are d blocked planes are overlapped is given by

$$\frac{\binom{m}{k_1}\binom{k_1-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_1}{k_2-d}}{\binom{m}{k_1}\binom{m-c_{(1/2)(\log N+1)}}{k_2-c_{(1/2)(\log N+1)}}} = \frac{\binom{k_1-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_1}{k_2-d}}{\binom{m-c_{(1/2)(\log N+1)}}{k_2-c_{(1/2)(\log N+1)}}}.$$

Symmetrically, we can prove that this probability is also given by

$$\frac{\binom{k_2-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_2}{k_1-d}}{\binom{m-c_{(1/2)(\log N+1)}}{k_1-c_{(1/2)(\log N+1)}}}.$$

QED

Based on the results of lemma 1, we can establish the following theorem concerning the blocking probability of a $\mathrm{VSPB}(N,m)$ network with odd number of stages.

Theorem 1: For $\mathrm{VSPB}(N,m)$, where $\log N$ is odd, see (2) shown at the bottom of the page, where

$$\begin{aligned}\Pr\Big(&n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}, \\ &b^{1,\ldots,(1/2)(\log N+1)} = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\ &= \Pr\left(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}\right) \\ &\quad \cdot \Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k_1 \,\Big|\, n_{(1/2)(\log N+1)} \\ &\qquad = c_{(1/2)(\log N+1)}\Big) \\ &\quad \times \Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k_2 \,\Big|\, n_{(1/2)(\log N+1)} \\ &\qquad = c_{(1/2)(\log N+1)}\Big). \end{aligned} \tag{3}$$

$$\begin{aligned}\Pr(&\text{blocking}) \\ &= 1 - \Pr(\text{nonblocking}) \\ &= 1 - \sum_{c_{(1/2)(\log N+1)}=0}^{\min(m-1,|I_{\text{odd}}|)} \sum_{k_1=c_{(1/2)(\log N+1)}}^{\min(m-1,|I_{\text{odd}}|)} \sum_{k_2=c_{(1/2)(\log N+1)}}^{\min(m-1,|O_{\text{odd}}|)} \sum_{d=\max\left(k_1+k_2-m+1, c_{(1/2)(\log N+1)}\right)}^{\min(k_1,k_2)} \frac{\binom{k_1-c_{(1/2)(\log N+1)}}{d-c_{(1/2)(\log N+1)}}\binom{m-k_1}{k_2-d}}{\binom{m-c_{(1/2)(\log N+1)}}{k_2-c_{(1/2)(\log N+1)}}} \\ &\quad \times \Pr\left(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\right)\end{aligned} \tag{2}$$

Here, probabilities $\Pr(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)})$, $\Pr\left(b^{1,\ldots,(1/2)(\log N+1)} = k_1 \left| n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}\right.\right)$ and $\Pr\left(b^{1,\ldots,(1/2)(\log N+1)} = k_2 \left| n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}\right.\right)$ are the functions of three basic parameters N, m, and r (the occupancy probability of an input/output link), and they can be evaluated efficiently as shown in Section III-C.

Proof: If there are d blocked planes overlapped in a $\mathrm{VSPB}(N,m)$ network given the events that $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$ and $b^{(1/2)(\log N+1),\ldots,\log N} = k_2 (c_{(1/2)(\log N+1)} \leq k_1, k_2 \leq m-1)$, the connection of the tagged path is not blocked if and only if there are at least one plane has neither half of the tagged SEs being blocked. In other words, it is not blocked if and only if $k_1 + k_2 - d < m$, or $d \geq \max(k_1 + k_2 - m + 1, c_{(1/2)(\log N+1)})$. Apparently, we have $d \leq \min(k_1, k_2)$. Thus, from (1) the probability that the connection of tagged path is not blocked provided the events $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$, and $b^{(1/2)(\log N+1),\ldots,\log N} = k_2$, is given by

$$\begin{aligned}
&\Pr\Big(\text{nonblocking}) n_{(1/2)(\log N+1)} \\
&\quad = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\quad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\
&= \Pr\Big(k_1 + k_2 - d < m \,\Big|\, n_{(1/2)(\log N+1)} \\
&\quad = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\quad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\
&= \sum_{d=\max\left(k_1+k_2-m+1, c_{(1/2)(\log N+1)}\right)}^{\min(k_1,k_2)} \\
&\quad \times \frac{\binom{k_1 - c_{(1/2)(\log N+1)}}{d - c_{(1/2)(\log N+1)}} \binom{m-k_1}{k_2-d}}{\binom{m - c_{(1/2)(\log N+1)}}{k_2 - c_{(1/2)(\log N+1)}}}.
\end{aligned}$$

Therefore, when $\log N$ is odd, the nonblocking probability for $\mathrm{VSPB}(N,m)$ is given by

$$\begin{aligned}
&\Pr(\text{nonblocking}) \\
&= \sum_{c_{(1/2)(\log N+1)}=0}^{\min(m-l,|I_{\mathrm{odd}}|)} \sum_{k_1=c_{(1/2)(\log N+1)}}^{\min(m-l,|I_{\mathrm{odd}}|)} \\
&\quad \times \sum_{k_2=c_{(1/2)(\log N+1)}}^{\min(m-l,|O_{\mathrm{odd}}|)} \sum_{d=\max(k_1+k_2-m+1,c_{(1/2)(\log N+1)})}^{\min(k_1,k_2)} \\
&\quad \times \frac{\binom{k_1 - c_{(1/2)(\log N+1)}}{d - c_{(1/2)(\log N+1)}} \binom{m-k_1}{k_2-d}}{\binom{m - c_{(1/2)(\log N+1)}}{k_2 c_{(1/2)(\log N+1)}}} \\
&\quad \times \Pr\Big(n_{(1/2)(\log N+1)} \\
&\qquad = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\qquad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big).
\end{aligned}$$

Based on the symmetry of the tagged path in Banyan networks with odd number of stages, we have

$$\begin{aligned}
&\Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k \,\Big|\, n_{(1/2)(\log N+1)} \\
&\quad = c_{(1/2)(\log N+1)}\Big) \\
&= \Pr\Big(b^{(1/2)(\log N+1),\ldots,\log N} = k \,\Big|\, n_{(1/2)(\log N+1)} \\
&\quad = c_{(1/2)(\log N+1)}\Big).
\end{aligned}$$

Thus, the probability $\Pr(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2)$ can be evaluated as

$$\begin{aligned}
&\Pr\Big(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\quad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\
&= \Pr(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}) \\
&\quad \times \Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k_1 \,\Big|\, n_{(1/2)(\log N+1)} \\
&\qquad = c_{(1/2)(\log N+1)}\Big) \\
&\quad \times \Pr\Big(b^{(1/2)(\log N+1),\ldots,\log N} = k_2 \,\Big|\, n_{(1/2)(\log N+1)} \\
&\qquad = c_{(1/2)(\log N+1)}\Big) \\
&= \Pr\left(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}\right) \\
&\quad \times \Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k_1 \,\Big|\, n_{(1/2)(\log N+1)} \\
&\qquad = c_{(1/2)(\log N+1)}\Big) \\
&\quad \times \Pr\Big(b^{1,\ldots,(1/2)(\log N+1)} = k_2 \,\Big|\, n_{(1/2)(\log N+1)} \\
&\qquad = c_{(1/2)(\log N+1)}\Big).
\end{aligned}$$

QED

The following corollary shows that the blocking probability derived above matches the condition of strictly nonblocking [10].

Corollary 1: When $\log N$ is odd, the blocking probability Pr(blocking) given in (2) becomes 0 if $m \geq (3/2)\sqrt{2N} - 1$

Proof: For $\mathrm{VSPB}(N,m)$ when $\log N$ is odd, the maximum number of conflicts with the tagged path occurs when all inputs in set I_i are destined for the outputs in set $O_{(\log N - i + 1)}$ and all outputs in O_i are originated from set $I_{(\log N - i + 1)}$ for $1 \leq i \leq (1/2)(\log N + 1)$. Note that all the inputs in set $I_{(1/2)(\log N+1)}$ are destined for the outputs in set $O_{(1/2)(\log N+1)}$ through the same shared paths, so only half of the total elements in these two sets can be counted. Thus, for the given events $n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}$, $b^{1,\ldots,(1/2)(\log N+1)} = k_1$, $b^{(1/2)(\log N+1),\ldots,\log N} = k_2$, and that d blocked planes are overlapped, the following relationship holds:

$$\begin{aligned}
k_1 + k_2 - d &\leq |I_{\mathrm{odd}}| + |O_{\mathrm{odd}}| - |I_{(1/2)}(\log N + 1)| \\
&= \left(\frac{3}{2}\right)\sqrt{2N} - 2.
\end{aligned}$$

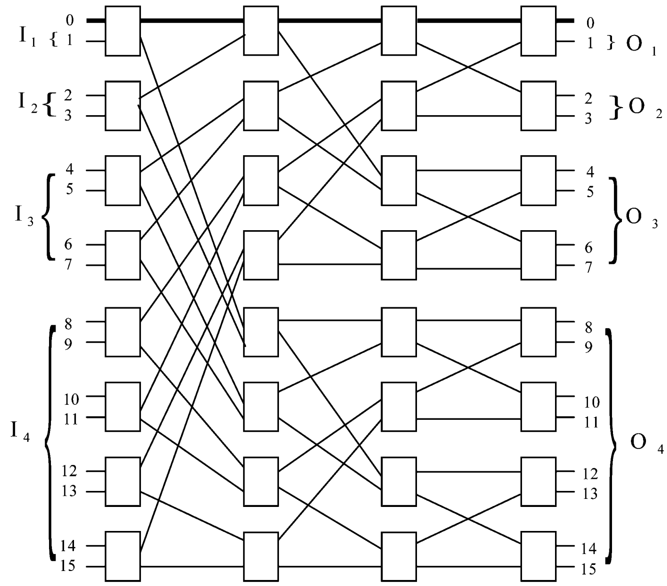

Fig. 3. A 16×16 Banyan network (even number of stages).

Therefore, for any given k_1 and k_2, if $m \ge (3/2)\sqrt{2N} - 2 + 1 = (3/2)\sqrt{2N} - 1$, we have $\Pr(k_1 + k_2 - d \ge m | n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2) = 0$. Thus, when $m \ge (3/2)\sqrt{2N} - 1$ the blocking probability is given by

$$\begin{aligned}
&\Pr(\text{blocking}) \\
&\quad = \sum_{c_{(1/2)(\log N+1)}=0}^{|I_{\text{odd}}|} \sum_{k_1=c_{(1/2)(\log N+1)}}^{|I_{\text{odd}}|} \sum_{k_2=c_{(1/2)(\log N+1)}}^{|O_{\text{odd}}|} \\
&\qquad \times \Pr\Big(k_1 + k_2 - d \ge m | \, n_{(1/2)(\log N+1)} \\
&\qquad\quad = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\qquad\quad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\
&\qquad \times \Pr\Big(n_{(1/2)(\log N+1)} \\
&\qquad\quad = c_{(1/2)(\log N+1)}, b^{1,\ldots,(1/2)(\log N+1)} \\
&\qquad\quad = k_1, b^{(1/2)(\log N+1),\ldots,\log N} = k_2\Big) \\
&\quad = 0.
\end{aligned}$$

QED

B. Blocking Probability When $\log N$ *is Even*

For the case when $\log N$ is even (please refer to Fig. 3), we further define input set I_{even} and output set O_{even} as

$$I_{\text{even}} = \bigcup_{i=1}^{(1/2)\log N} I_i, \quad O_{\text{even}} = \bigcup_{i=1}^{(1/2)\log N} O_i.$$

For a $\text{VSPB}(N, m)$ network where $\log N$ is even, the tagged path is also composed of two symmetric halves, each of which has $(1/2)\log N$ stages. Note that under the crosstalk-free constraint, a plane of a $\text{VSPB}(N, m)$ network can be blocked in the first half of the tagged SEs (from stage 1 to stage $(1/2)\log N$) or/and in the second half of the tagged SEs (from stage $(1/2)\log N + 1$ to stage $\log N$). Then the maximum number of conflicts with the first half of the tagged SEs and second half of the tagged SEs is determined by the number of connections from set I_{even} and the number of connections destined for set O_{even}, respectively. If a plane of a $\text{VSPB}(N, m)$ network is blocked in both the first and second halves of tagged SEs (i.e., a plane blocked in the first half of tagged SEs shares a common plane blocked in the second half of tagged SEs), this blocked plane is said to be overlapped. The following lemma concerns the number of overlapped planes for a $\text{VSPB}(N, m)$ network when $\log N$ is even.

Lemma 2: Let $t_{i,i+1}$ be the number of connections passing through both the ith and the $(i+1)$th tagged SEs. Given the events $n_{(1/2)\log N} = c_{(1/2)\log N}$, $n_{(1/2)\log N+1} = c_{(1/2)\log N+1}$, $t_{(1/2)\log N,(1/2)\log N+1} = l$, $b^{1,\ldots,(1/2)\log N} = k_1$ and $b^{(1/2)\log N+1,\ldots,\log N} = k_2$ in a $\text{VSPB}(N, m)$ network when $\log N$ is even $(l \le k_1, k_2 \le m-1)$, the probability that there are d blocked planes overlapped is given by

$$\begin{aligned}
&\Pr\Big(d \text{ blocked planes overlapped} \,\big|\, n_{(1/2)\log N} \\
&\quad = c_{(1/2)\log N}, \; n_{(1/2)\log N+1} = c_{(1/2)\log N+1}, \\
&\qquad t_{(1/2)\log N,(1/2)\log N+1} = l, b^{1,\ldots,(1/2)\log N} \\
&\quad = k_1, b^{(1/2)\log N+1,\ldots,\log N} = k_2\Big) \\
&\quad = \frac{\binom{k_1-l}{d-l}\binom{m-k_1}{k_2-d}}{\binom{m-l}{k_2-l}} = \frac{\binom{k_2-l}{d-l}\binom{m-k_2}{k_1-d}}{\binom{m-l}{k_1-l}}.
\end{aligned}$$

Proof: The lemma can be proved in a similar way as that of Lemma 1.

Based on the results in lemma 2, we can establish the following theorem regarding the blocking probability of a $\text{VSPB}(N, m)$ network with even number of stages.

Theorem 2: For a $\text{VSPB}(N, m)$ network, where $\log N$ is even

$$\begin{aligned}
&\Pr(\text{blocking}) \\
&\quad = 1 - \Pr(\text{nonblocking}) \\
&\quad = 1 - \sum_{c_{(1/2)\log N}=0}^{\min(m-1,|I_{\text{even}}|)} \; \sum_{c_{(1/2)\log N+1}=0}^{\min(m-1,|O_{\text{even}}|)} \\
&\qquad \times \sum_{l=0}^{\min\left(c_{(1/2)\log N}, c_{(1/2)\log N+1}\right)} \\
&\qquad \times \sum_{k_1=c_{(1/2)\log N}}^{\min(m-1,|I_{\text{even}}|)} \; \sum_{k_2=c_{(1/2)\log N+1}}^{\min(m-1,|O_{\text{even}}|)} \\
&\qquad \times \sum_{d=\max(k_1+k_2-m+1,l)}^{\min(k_1,k_2)} \frac{\binom{k_1-l}{d-l}\binom{m-k_1}{k_2-d}}{\binom{m-l}{k_2-l}} \\
&\qquad \times \Pr(n_{(1/2)\log N} = c_{(1/2)\log N}, n_{(1/2)\log N+1} \\
&\quad = c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1} \\
&\quad = l, b^{1,\ldots,(1/2)\log N} = k_1, b^{(1/2)\log N+1,\ldots,\log N} = k_2)
\end{aligned} \tag{4}$$

where

$$\begin{aligned}\Pr\Big(&n_{(1/2)\log N}=c_{(1/2)\log N}, n_{(1/2)\log N+1}\\&=c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1}=l,\\&b^{1,\ldots,(1/2)\log N}=k_1, b^{(1/2)\log N+1,\ldots,\log N}=k_2\Big)\\=\Pr\big(&n_{(1/2)\log N}=c_{(1/2)\log N}, n_{(1/2)\log N+1}\\&=c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1}=l\big)\\&\times\Pr\Big(b^{1,\ldots,(1/2)\log N}=k_1\,|n_{(1/2)\log N}\\&\quad=c_{(1/2)\log N}\Big)\\&\times\Pr\Big(b^{1,\ldots,(1/2)\log N}=k_2\,|n_{(1/2)\log N}\\&\quad=c_{(1/2)\log N+1}\Big).\end{aligned}\tag{5}$$

Here, probabilities $\Pr(n_{(1/2)\log N}=c_{(1/2)\log N}, n_{(1/2)\log N+1}=c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1}=l)$, $\Pr(b^{1,\ldots,(1/2)\log N}=k_1 \mid n_{(1/2)\log N}=c_{(1/2)\log N})$ and $\Pr(b^{1,\ldots,(1/2)\log N}=k_2 \mid n_{(1/2)\log N}=c_{(1/2)\log N+1})$ are also the functions of three basic parameters N, m and r and they can be evaluated efficiently as shown in Section III-C.

Proof: The proof of this theorem is similar to the proof of Theorem 1.

The following corollary shows that when $\log N$ is even, the blocking probability we derived also agrees with the deterministic condition for strictly nonblocking.

Corollary 2: When $\log N$ is even, the blocking probability Pr(blocking) given in (4) becomes 0 if $m\geq 2\sqrt{N}-1$.

Proof: The proof for the Corollary is the same as that for Corollary 1, except that when $\log N$ is even, the scenario of the maximum number of conflicts with the tagged path is when all inputs in set I_i are destined for the outputs in set $O_{(\log N-i+1)}$ and all outputs in O_i are originated from set $I_{(\log N-i+1)}$ for $1\leq i\leq(1/2)\log N$. Thus, for given events $b^{1,\ldots,(1/2)\log N}=k_1$, $b^{(1/2)\log N+1,\ldots,\log N}=k_2$ and that d blocked planes are overlapped, the following inequality always holds:

$$k_1+k_2-d\leq|I_{\text{even}}|+|O_{\text{even}}|=2(\sqrt{N}-1).$$

QED

C. Calculation of $\Pr(b^{1,\ldots,i,i+1}=k \mid n_{i+1}=c_{i+1})$

The main problem remains unsolved is how to evaluate the probability $\Pr(b^{1,\ldots,(1/2)(\log N+1)}=k \mid n_{(1/2)(\log N+1)}=c_{(1/2)(\log N+1)})$ (when $\log N$ is odd) and the probability $\Pr(b^{1,\ldots,(1/2)\log N}=k \mid n_{(1/2)\log N}=c_{(1/2)\log N})$ (when $\log N$ is even). In this section, we first calculate the probability $\Pr(b^{1,2}=k \mid n_2=c_2)$, then we will show that the probability $\Pr(b^{1,\ldots,i,i+1}=k \mid n_{i+1}=c_{i+1})$ can be calculated from the model of $\Pr(b^{1,\ldots,i}=g \mid n_i=c_i)$. Based on the results, a novel approach is developed to calculate the probabilities $\Pr(b^{1,\ldots,(1/2)(\log N+1)}=k \mid n_{(1/2)(\log N+1)}=c_{(1/2)(\log N+1)})$ and $\Pr(b^{1,\ldots,(1/2)\log N}=k \mid n_{(1/2)\log N}=c_{(1/2)\log N})$, recursively. The evaluation of probabilities $\Pr(n_{(1/2)(\log N+1)}=c_{(1/2)(\log N+1)})$ in (3) and $\Pr(n_{(1/2)\log N}=c_{(1/2)\log N}, n_{(1/2)\log N+1}=c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1}=l)$ in (5) will also be discussed in the rest of this section.

1) Calculation of $\Pr(b^{1,2}=k \mid n_2=c_2)$: For a $\text{VSPB}(N,m)$ network, we have

$$\begin{aligned}\Pr\left(b^{1,2}=k\,|n_2=c_2\right)=&\sum_{c_1=\max(0,c_2-|I_2|)}^{\min(m-1,|I_1|)}\Pr(n_1=c_1\,|n_2=c_2)\\&\times\Pr\left(b^{1,2}=k\,|n_1=c_1,n_2=c_2\right)\end{aligned}$$

where the evaluation of $\Pr(n_1=c_1 \mid n_2=c_2)$ will be further discussed in Section III). Let $t_{i,i+1}$ denotes the number of connections passing through both the ith and the $(i+1)$th tagged SEs, $\Pr(b^{1,2}=k \mid n_1=c_1,n_2=c_2)$ can be expressed as

$$\begin{aligned}\Pr&\left(b^{1,2}=k\,|n_1=c_1,n_2=c_2\right)\\&=\sum_{l=\max(0,c_2-|I_2|)}^{\min\{c_1,c_2\}}\Pr(t_{1,2}=l\,|n_1=c_1,n_2=c_2)\\&\quad\times\Pr\left(b^{1,2}=k\,|t_{1,2}=l,n_1=c_1,n_2=c_2\right).\end{aligned}$$

The calculation of probability $\Pr(t_{1,2}=l \mid n_1=c_1,n_2=c_2)$ will be provided in Section III), and the evaluation of probability $\Pr(b^{1,2}=k \mid t_{1,2}=l,n_1=c_1,n_2=c_2)$ is presented in the following lemma.

Lemma 3: Given events $n_1=c_1$, $n_2=c_2$, and $t_{1,2}=l$ in a $\text{VSPB}(N,m)$ network, the probability that $b^{1,2}=k$ (where $\max(c_1,c_2)\leq k\leq\min(m-1,c_1+c_2-l)$) is given by

$$\begin{aligned}\Pr&\left(b^{1,2}=k\,|t_{1,2}=l,n_1=c_1,n_2=c_2\right)\\&=\frac{\binom{\max(c_1,c_2)-l}{c_1+c_2-k-l}\binom{m-\max(c_1,c_2)}{k-\max(c_1,c_2)}}{\binom{m-l}{\min(c_1,c_2)-l}}.\end{aligned}$$

Proof: Under the crosstalk-free constraint, all connections passing through a tagged SE must fall in distinct planes, and these planes will be blocked for the connection of the tagged path. Given events $n_1=c_1$, $n_2=c_2$, and $t_{1,2}=l$ in a $\text{VSPB}(N,m)$ network under random routing, we have total $\binom{m}{\max(c_1,c_2)}\binom{m-l}{\min(c_1,c_2)-l}$ ways to select $\max(c_1,c_2)$ blocked planes corresponding to $\max(c_1,c_2)$ connections and $\min(c_1,c_2)$ blocked planes corresponding to $\min(c_1,c_2)$ connections. Thus, a total of k blocked planes ($\max(c_1,c_2)\leq k\leq\min(m-1,c_1+c_2-l)$) can be constructed as follows (see Fig. 4): there are $\binom{m}{\max(c_1,c_2)}$ ways to select $\max(c_1,c_2)$ blocked planes and $\binom{\max(c_1,c_2)-l}{c_1+c_2-k-l}$ ways to select $\max(c_1,c_2)+\min(c_1,c_2)-k-l=c_1+c_2-k-l$ overlapped planes that are blocked by both the connections from $\max(c_1,c_2)-l$ and the connections from $\min(c_1,c_2)-l$. Finally, the $k-\max(c_1,c_2)$ planes blocked by the rest of $\min(c_1,c_2)-l-(c_1+c_2-k-l)=k-\max(c_1,c_2)$ connections must be selected from $m-\max(c_1,c_2)$ planes, and we have $\binom{m-\max(c_1,c_2)}{k-\max(c_1,c_2)}$ choices. Therefore, given events

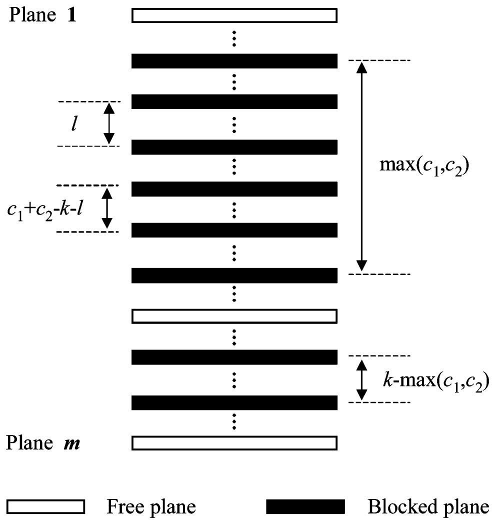

Fig. 4. VSPB (N,m) with $c1$ connections passing through the first tagged SE, $c2$ connections passing through the second tagged SE, among them l connections passing through both tagged SEs, and there are a total of k planes $(\max(c_1,c_2) \le k \le kc_1 + c_2 - l)$ blocked.

$n_1 = c_1, n_2 = c_2$, and $t_{1,2} = l$ in a VSPB(N,m) network, the probability that k planes are blocked is

$$\frac{\binom{m}{\max(c_1,c_2)}\binom{\max(c_1,c_2)-l}{c_1+c_2-k-l}\binom{m-\max(c_1,c_2)}{k-\max(c_1,c_2)}}{\binom{m}{\max(c_1,c_2)}\binom{m-l}{\min(c_1,c_2)-l}} = \frac{\binom{\max(c_1,c_2)-l}{c_1+c_2-k-l}\binom{m-\max(c_1,c_2)}{k-\max(c_1,c_2)}}{\binom{m-l}{\min(c_1,c_2)-l}}.$$

QED

2) Calculation of $\Pr\left(b^{1,\ldots,i,i+1} = k \mid n_{i+1} = c_{i+1}\right)$ *Based on* $\Pr\left(b^{1,\ldots,i} = g \mid n_i = c_i\right)$: To show that $\Pr\left(b^{1,\ldots,i,i+1} = k \mid n_{i+1} = c_{i+1}\right)$ can be evaluated recursively, we first express it as

$$\begin{aligned}\Pr&\left(b^{1,\ldots,i,i+1} = k \,|n_{i+1} = c_{i+1}\right)\\ &= \sum_{c_i=\max(0,c_{i+1}-|I_{i+1}|)}^{\min(m-1,|I_1|+\cdots+|I_i|)} \Pr\left(n_i = c_i \,|n_{i+1} = c_{i+1}\right)\\ &\quad\cdot \Pr\left(b^{1,\ldots,i,i+1} = k \mid n_i = c_i, n_{i+1} = c_{i+1}\right).\end{aligned}$$

The evaluation of $\Pr\left(n_i = c_i \mid n_{i+1} = c_{i+1}\right)$ will be provided in Section III), and $\Pr\left(b^{1,\ldots,i,i+1} = k \mid n_i = c_i, n_{i+1} = c_{i+1}\right)$ is given by

$$\begin{aligned}\Pr&\left(b^{1,\ldots,i,i+1} = k \mid n_i = c_i, n_{i+1} = c_{i+1}\right)\\ &= \sum_{g=c_i}^{\min(m-1,|I_1|+\cdots+|I_{i-1}|+\min(c_i,|I_i|))} \Pr\left(b^{1,\ldots,i} = g \,|n_i = c_i\right)\\ &\quad\times \Pr\left(b^{1,\ldots,i,i+1} = k \,\middle|\, b^{1,\ldots,i} = g,\right.\\ &\qquad\left. n_i = c_i, n_{i+1} = c_{i+1}\right).\end{aligned}$$

The probability $\Pr\left(b^{1,\ldots,i,i+1} = k \mid b^{1,\ldots,i} = g, n_i = c_i, n_{i+1} = c_{i+1}\right)$ can be evaluated as

$$\begin{aligned}\Pr&\left(b^{1,\ldots,i,i+1} = k \,\middle|\, b^{1,\ldots,i} = g,\, n_i = c_i, n_{i+1} = c_{i+1}\right)\\ &= \sum_{l=\max(0,c_{i+1}-|I_{i+1}|)}^{\min(c_i,c_{i+1})} \Pr\left(t_{i,i+1} = l \,|n_i = c_i, n_{i+1} = c_{i+1}\right)\\ &\quad\times \Pr\left(b^{1,\ldots,i,i+1} = k \,\middle|\, b^{1,\ldots,i} = g,\right.\\ &\qquad\left. t_{i,i+1} = l,\ n_i = c_i, n_{i+1} = c_{i+1}\right).\end{aligned}$$

where the evaluation of $\Pr\left(t_{i,i+1} = l \mid n_i = c_i, n_{i+1} = c_{i+1}\right)$ will be given in Section III), and the probability $\Pr\left(b^{1,\ldots,i,i+1} = k \mid b^{1,\ldots,i} = g, t_{i,i+1} = l, n_i = c_i, n_{i+1} = c_{i+1}\right)$ for $\max(g, c_{i+1}) \le k \le \min(m-1, g + c_{i+1} - l)$ is given by the following formula which is obtained by an approach similar to that of lemma 3

$$\begin{aligned}\Pr&\left(b^{1,\ldots,i,i+1} = k \,\middle|\, b^{1,\ldots,i} = g,\right.\\ &\quad\left. t_{i,i+1} = l, n_i = c_i, n_{i+1} = c_{i+1}\right)\\ &\quad= \frac{\binom{\max(g,c_{i+1})-l}{g+c_{i+1}-k-l}\binom{m-\max(g,c_{i+1})}{k-\max(g,c_{i+1})}}{\binom{m-l}{\min(g,c_{i+1})-l}}.\end{aligned}$$

The above process clearly indicates that the evaluation of the probability $\Pr(b^{1,\ldots,i,i+1} = k \mid n_{i+1} = c_{i+1})$ is totally based on the result of $\Pr(b^{1,\ldots,i} = g \mid n_i = c_i)$. Therefore, we have shown above an efficient approach for computing the probability $\Pr\left(b^{1,\ldots,(1/2)(\log N+1)} = k \,\middle|\, n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)}\right)$ (when $\log N$ is odd) and the probability $\Pr(b^{1,\ldots,(1/2)\log N} = k \mid n_{(1/2)\log N} = c_{(1/2)\log N})$ (when $\log N$ is even) stage by stage recursively.

3) Calculation of $\Pr(t_{i,i+1} = l \mid n_i = c_i, n_{i+1} = c_{i+1})$ *and* $\Pr(n_i = c_i \mid n_{i+1} = c_{i+1})$: To calculate $\Pr(t_{i,i+1} = l \mid n_i = c_i, n_{i+1} = c_{i+1})$, we first use the following relationship:

$$\begin{aligned}\Pr&(t_{i,i+1} = l \,|n_i = c_i, n_{i+1} = c_{i+1})\\ &= \frac{\Pr(n_i = c_i, n_{i+1} = c_{i+1}, t_{i,i+1} = l)}{\Pr(n_i = c_i, n_{i+1} = c_{i+1})}\\ &= \frac{\Pr(n_i = c_i, n_{i+1} = c_{i+1}, t_{i,i+1} = l)}{\sum_{s=\max(0,c_{i+1}-|I_{i+1}|)}^{\min(c_i,c_{i+1})} \Pr(n_i = c_i, n_{i+1} = c_{i+1}, t_{i,i+1} = s)}.\end{aligned} \tag{6}$$

Since the correlation between n_i and n_{i+1} is determined by the connections passing through both the ith and the $(i+1)$th tagged SEs, we have

$$\begin{aligned}\Pr&(n_i = c_i, n_{i+1} = c_{i+1}, t_{i,i+1} = s)\\ &= \Pr(t_{i,i+1} = s, n_i = c_i)\\ &\quad\times \Pr(n_{i+1} = c_{i+1} | t_{i,i+1} = s, n_i = c_i)\\ &= \Pr(t_{i,i+1} = s, n_i = c_i)\\ &\quad\times \Pr(n_{i+1} = c_{i+1} | t_{i,i+1} = s)\end{aligned} \tag{7}$$

where $\Pr(t_{i,i+1} = s, n_i = c_i)$ is given by (8) shown at the bottom of the page and $\Pr(n_{i+1} = c_{i+1} \mid t_{i,i+1} = s)$ can be evaluated as

$$\Pr(n_{i+1} = c_{i+1} | t_{i,i+1} = s) = \sum_{f=c_{i+1}-s}^{|I_{i+1}|} \binom{|I_{i+1}|}{f} \cdot r^f \cdot (1-r)^{|I_{i+1}|-f} \times \frac{\binom{\sum_{j=1}^{\log N-i} |O_j| - s}{c_{i+1}-s} \binom{\sum_{j=\log N-i+1}^{\log N} |O_j|}{f-c_{i+1}+s}}{\binom{N-1-s}{f}} \tag{9}$$

where r is the occupancy probability of an input (output) link.

To calculate $\Pr(n_i = c_i \mid n_{i+1} = c_{i+1})$, we first express it as

$$\Pr(n_i = c_i | n_{i+1} = c_{i+1}) = \frac{\Pr(n_i = c_i, n_{i+1} = c_{i+1})}{\Pr(n_{i+1} = c_{i+1})} \tag{10}$$

where $\Pr(n_{i+1} = c_{i+1})$ is given by

$$\Pr(n_{i+1} = c_{i+1}) = \sum_{c_i=\max(0, c_{i+1}-|I_{i+1}|)}^{|I_1|+\cdots+|I_i|} \Pr(n_i = c_i, n_{i+1} = c_{i+1}). \tag{11}$$

The probability $\Pr(n_i = c_i, n_{i+1} = c_{i+1})$ in (10) and (11) can be evaluated in the same way, as shown in (6)–(9).

The above process indicates that we can actually calculate the probability $\Pr(n_{(1/2)(\log N+1)} = c_{(1/2)(\log N+1)})$ in (3) based on (11) and calculate the probability $\Pr(n_{(1/2)\log N} = c_{(1/2)\log N}, n_{(1/2)\log N+1} = c_{(1/2)\log N+1}, t_{(1/2)\log N,(1/2)\log N+1} = l)$ in (5) based on (7)–(9).

IV. Experimental Results

An experiment was performed to verify our theoretical model. We developed a network simulator, which employed the random routing strategy to determine the blocking probability of a VSPB network under the same conditions used to develop the theoretical model. The network simulator consists of two major modules: the request pattern generator and the request router. The request pattern generator randomly generates a request pattern consisting of a list of feasible connection requests based on r (the occupancy probability of an input/output link) and N (the number of input/output ports). The request router attempts to route these feasible connection requests through the network using the random routing algorithm, with which the request router randomly chooses one of the planes that can be used by a feasible connection request to establish the connection. If no plane is available to satisfy the request for the tagged path, the connection request pattern is recorded as a blocked connection pattern. The blocking probability is then estimated by the ratio between the number of blocked connection patterns and the total number of connection patterns generated. During the simulation, a certain workload is maintained for each trial, which is defined as the probability that an input (or an output) link is occupied.

A. Comparison Between Theoretical and Simulated Results On Blocking Probability

Two network configurations with $N = 16$ and $N = 32$ are adopted in the simulation to verify the proposed analytical model. For each configuration, blocking probability is measured for both the theoretical model and the experiment. A comparison between the theoretical results and the simulation results for $N = 16$ and $N = 32$ is summarized in Figs. 5 and 6, respectively, which indicates clearly that our theoretical model is very efficient for estimating the blocking probabilities of VSPB networks under the random routing strategy.

It can also be seen that our theoretical model agrees well with the condition of strictly nonblocking as demonstrated in the study of [10]. For the network configuration $N = 16$, the blocking probability goes to zero at $m = 2\sqrt{N} - 1 = 7$. For $N = 32$, the blocking probability goes to zero at $m = (3/2)\sqrt{2N} - 1 = 11$. It is interesting to note in Figs. 5(b) and 6(b) that although the blocking probabilities corresponding to different numbers of planes increase monotonically with the increase of workload, their sensitivities to the workload variation are different. When the workload is increased from 0.65 to 0.95, the blocking probability is increased from 0.55 to 0.83, from 0.015 to 0.055, from 0.28 to 0.58, and from 0.007 to 0.04, for each of the cases VSPB(16,2), VSPB(16,4), VSPB(32,3), and VSPB(32,5), respectively.

The above results show that our model can accurately describe the blocking behavior of VSPB networks under random routing. The results in Figs. 5 and 6 further indicate that it is possible for us to dramatically reduce the hardware cost (or the

$$\Pr(t_{i,i+1} = s, n_i = c_i) = \sum_{f=c_i}^{2^i-1} \binom{\sum_{j=1}^{i} |I_j|}{f} \cdot r^f \cdot (1-r)^{\sum_{j=1}^{i} |I_j| - f} \times \frac{\binom{\sum_{j=1}^{\log N-i} |O_j|}{s} \binom{|O_{\log N-i+1}|}{c_i - s} \binom{N-1-\sum_{j=1}^{\log N-i+1} |O_j|}{f-c_i}}{\binom{N-1}{f}} \tag{8}$$

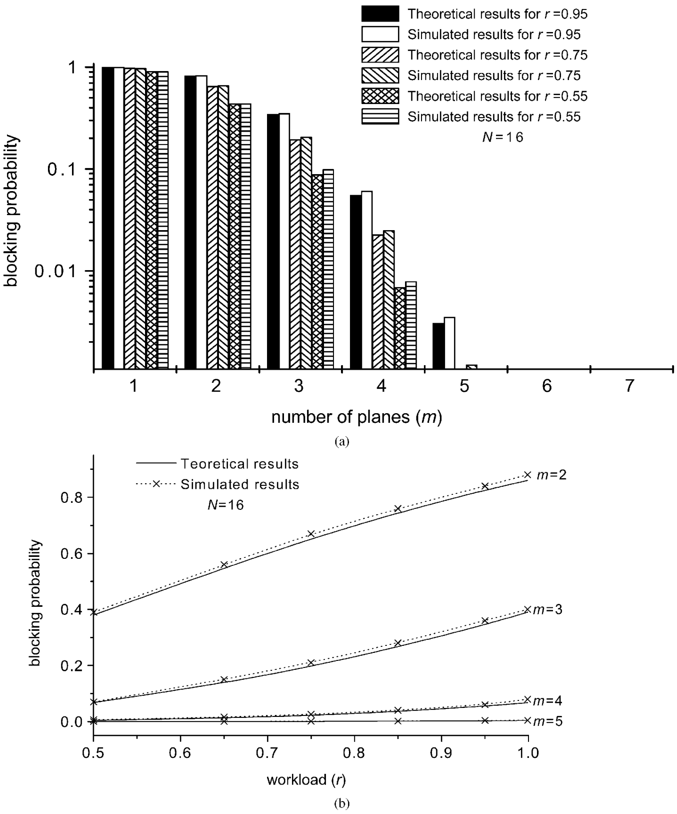

Fig. 5. Blocking probability of $\mathrm{VSPB}(16, m)$ network. (a) Blocking probability versus m with $r = \{0.55, 0.75, 0.95\}$. (b) Blocking probability versus r with $m = \{2, 3, 4, 5\}$.

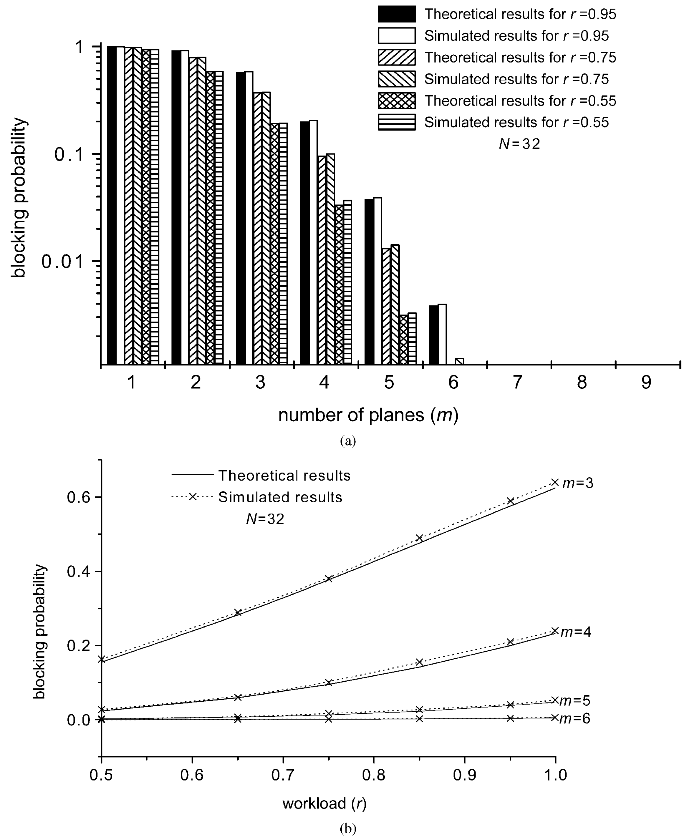

Fig. 6. Blocking probability of $\mathrm{VSPB}(32, m)$ network. (a) Blocking probability versus m with $r = \{0.55, 0.75, 0.95\}$. (b) Blocking probability versus r with $m = \{3, 4, 5, 6\}$.

number of planes) with only little sacrifice in blocking probability of a network. Thus, our model can guide the network designers to estimate the blocking probability and to display the tradeoff between hardware cost and blocking probability for a VSPB network.

B. Hardware Cost Versus Blocking Probability

When the workload $r = 0.9$ for the VSPB networks of different sizes, the minimum numbers of planes estimated by our analytical model for different requirements of blocking probability (denoted by BP hereafter) are summarized in Table I. For comparison, we also show in Table I the minimum numbers of planes corresponding to the condition of strictly nonblocking $(\mathrm{BP} = 0)$. The results in Table I indicate that, for networks with larger sizes, the hardware cost required to achieve the nonblocking condition is considerably high, and it can be significantly reduced by allowing an almost negligible blocking probability. For example, in a network with $N = 1024$ and an upper limit of blocking probability 0.1% (i.e., $\mathrm{BP} < 0.1\%$), the minimum number of planes required to achieve the nonblocking condition is $m = 2\sqrt{N} - 1 = 63$, while the minimum number of planes estimated by our model is only 11 for $r = 0.9$. Therefore, we can save $(63 - 11)/63 \cong 82\%$ of the hardware cost, while keeping a very low blocking probability $(\mathrm{BP} < 0.1\%)$. We can also observe from Table I that for a given workload, the hardware cost estimated by our model is not sensitive to the requirements of blocking probability. For the network with $N = 1024$ and $r = 0.9$, the minimum number of planes

TABLE I
MINIMUM NUMBERS OF PLANES FOR $r = 0.9$ AND DIFFERENT REQUIREMENTS OF BLOCKING PROBABILITY

N	16	32	64	128	256	512	1024
$BP = 0$	7	11	15	23	31	47	63
$BP < 0.1\%$	6	7	8	9	10	11	11
$BP < 1\%$	5	6	7	8	8	9	10
$BP < 5\%$	4	5	6	7	7	8	8

TABLE II
MINIMUM NUMBERS OF PLANES FOR BP $< 1\%$ AND DIFFERENT WORKLOADS

N	16	32	64	128	256	512	1024
$BP = 0$	7	11	15	23	31	47	63
r=0.55	4	5	6	6	7	7	8
r=0.75	5	6	6	7	8	8	9
r=0.95	5	6	7	8	9	9	10

estimated by the model is 10 for the requirement of $\mathrm{BP} < 1\%$ and is 11 for the requirement of $\mathrm{BP} < 0.1\%$, both of which are much less than the number of planes (i.e., 63) required by the condition of strictly nonblocking.

Table II shows the minimum number of planes estimated by our model under different workloads when the blocking probability is required to be less than 1%, which clearly demonstrates that the hardware cost in each case estimated by our model is far less than that determined by nonblocking condition for large networks. For example, in the network with $N = 512$, the minimum number of planes required to achieve the nonblocking condition is $m = (3/2)\sqrt{2N} - 1 = 47$, while the minimum number of planes estimated by our model is only 8 for $r = 75\%$ and $\mathrm{BP} < 1\%$. Therefore, $(47-8)/47 \cong 83\%$ of the hardware cost can be saved, while a very low blocking probability is guaranteed $(\mathrm{BP} < 1\%)$.

We can also find from Table II that for a given requirement on blocking probability, the hardware cost is also not sensitive to the variation of workload. For the network $\mathrm{VSPB}(512, m)$ with the requirement of $\mathrm{BP} < 1\%$, the minimum number of planes estimated by the analytical model is 8 for $r = 0.75$ and 9 for $r = 0.95$, and all these numbers of planes are far less than the 47 planes determined by the nonblocking condition.

V. Conclusion

In this paper, the blocking behavior of vertically stacked photonic Banyan (VSPB) networks under the random routing strategy is studied, in which an analytical model is developed. By fully exploring the property of symmetry for Banyan network structures and the recursive characteristics in calculations, the model can evaluate the blocking probability of a VSOB network efficiently. A theoretical analysis along with extensive simulations by a network simulator are conducted, which verify that our model can accurately describe the blocking behavior of a VSPB network under random routing, while agreeing with the conditions of strictly nonblocking VSPB networks. The model reveals the inherent relationship between blocking probability and switch hardware cost in terms of the number of planes, in which an effective tool for the design of DC-based photonic switches is provided by initiating a graceful compromise between the desirable hardware cost and the guaranteed performance in terms of blocking probability. We conclude that the hardware cost of a VSPB network can be dramatically reduced if a predictable and almost negligible nonzero blocking probability is allowed. This characteristic makes the adoption of VSPB networks very attractive in most practical applications. Note that the crosstalk-free constraint adopted in this paper may not be necessary for some applications that are not sensitive to crosstalk. For a VSPB network without the crosstalk constraint, the approach proposed in this paper can be extended easily to model the blocking probability of the network by considering only the link-blocking instead of crosstalk-blocking in analysis. The approach of probabilistic analysis in this paper was developed mainly for optical switches built on the $\log_2 N$ Banyan networks. One of our future research directions is to extend the approach in this paper to analyze the performance of optical switches built on general $\log_d N$ networks $(d \geq 2)$. It is envisioned, however, that the approach developed this paper cannot be extended in a straightforward way to model the blocking probability of optical switches built on general $\log_d N$ networks, so a new and deliberate study is deserved. Note that the approach proposed in this paper can only be used to calculate the blocking probability of VSPB networks recursively, so our another future research direction is to develop a close-form formula for the blocking probability of VSPB networks, such that the tradeoff between the blocking probability and the network parameters (including network size, number of parallel planes, and offered load) can be expressed analytically.

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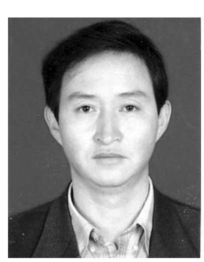

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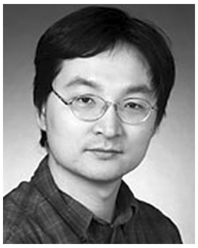

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